Advanced Mechanics: Final Exam

Semester Ib 2021-2022

January 27th, 2022

Please note the following rules:

- you are not allowed to use the book or the lecture notes, nor other notes or books
- you have 2 hours to complete the exam
- please write your student number on each paper sheet you hand in
- please use a separate paper sheet for each problem
- please raise your hand for more paper or to ask a question
- some useful equations are provided on the next page

Points for each problem:

- Problem 1: *5 points*
- Problem 2: *10 points*
- Problem 3: *20 points*
- Problem 4: *15 points*

Useful equations

- Generalised momentum:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{0.1}$$

- Hamiltonian:

$$H(q_j, p_j; t) = \dot{q}_i p_i - L(q_j, \dot{q}_j; t) \tag{0.2}$$

- Hamilton's equations:

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j \,, \tag{0.3}$$

$$\frac{\partial H}{\partial p_j} = \dot{q}_j \tag{0.4}$$

- Principal moments of inertia:

$$\mathrm{I_{xx}} = \int \mathrm{dm}\,(y^2 + z^2)\,, \tag{0.5}$$

$$\mathrm{I_{yy}} = \int \mathrm{dm}\,(x^2 + z^2)\,, \tag{0.6}$$

$$\mathrm{I_{zz}} = \int \mathrm{dm}\,(x^2 + y^2)\,. \tag{0.7}$$

- Infinitesimal (time-like) interval in special relativity:

$$ds^2 = c^2\,dt^2 - (d\mathbf{x})^2 \tag{0.8}$$

- Volume element in spherical coordinates:

$$dV = r^2\,dr\,\sin\theta\,d\theta\,d\phi \tag{0.9}$$

- Integrals of sinusoidal functions (for problem 4):

$$\int_0^{2\pi} \sin^2\phi\;d\phi = \pi\,, \tag{0.10}$$

$$\int_0^{2\pi} \cos^2\phi\;d\phi = \pi\,, \tag{0.11}$$

$$\int_0^{\pi} \cos^2\theta\,\sin\theta\;d\theta = \frac{2}{3}\,. \tag{0.12}$$

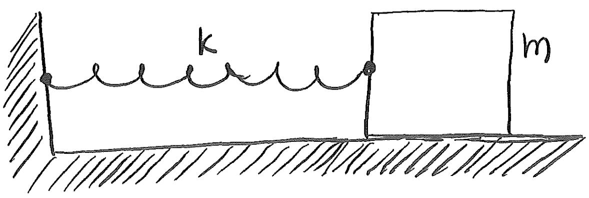

Figure 1.

Problem 1

Consider an harmonic oscillator with a spring constant k and mass m as in Fig. 1. The mass slides without friction on a horizontal plane. Compute the Hamiltonian and the explicit Hamilton's equations of motion.
Note: for clarity, please don't forget to define your coordinates of choice.
[5 points]

Problem 2

Consider the relativistic action written as an invariant quantity using the *proper time*

$$S = -kc \int_{\tau_1}^{\tau_2} d\tau\,, \tag{0.13}$$

where k is a constant and c is the speed of light. Using the definition of proper time (*time measured by a clock in its own rest frame*), show that the relativistic Lagrangian for a free relativistic particle of mass m in an arbitrary frame can be written as

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}}\,. \tag{0.14}$$

Please make sure you explain the steps in your derivation.
Hint: in order to determine k, expand the Lagrangian for small velocities (v/c much smaller than 1) and compare with the non-relativistic Lagrangian of a free particle.
[10 points]

Problem 3

Consider a particle of mass m moving under the effect of a central force (potential V(r)).

(3a) Compute the Lagrangian in spherical coordinates ($\{r,\, \theta,\, \phi\}$ as in Fig. 2) verifying that it is given by:

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\;\dot{\phi}^2\right) - V(r)\,. \tag{0.15}$$

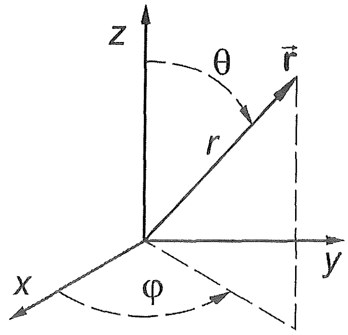

Figure 2. From cartesian to spherical coordinates: $x = r\,\sin\theta\cos\phi$, $y = r\,\sin\theta\sin\phi$, $z = r\cos\theta$, where $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$.

[10 points]

(3b) Compute the generalised momenta $(p_r,\, p_\theta,\, p_\phi)$, and the Hamiltonian $H(r,\, \theta,\, \phi,\, p_r,\, p_\theta,\, p_\phi)$.
[5 points]

(3c) Compute the explicit Hamilton's equations of motion.
[5 points]

Problem 4

Consider a sphere of radius R and uniformly distributed mass m (see Fig. 3). Determine the principal moments of inertia (I_{xx}, I_{yy} and I_{zz}) with respect to the center of mass of the sphere.
Please note: you should take advantage of symmetry arguments in order to shorten your calculations: how do I_{yy} and I_{zz} relate to I_{xx}?
[15 points]

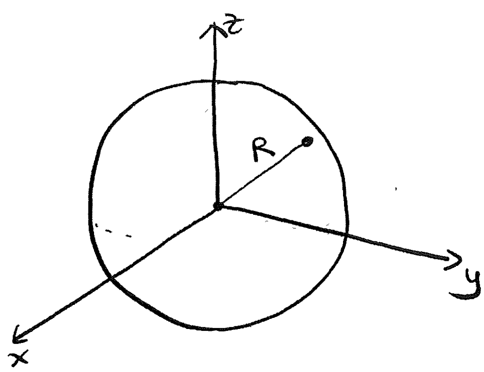

Figure 3.